J. Fluid Mech. (1979), *vol.* 91, *part* 1, *pp.* 181–190
Printed in Great Britain

On the Korteweg–de Vries equation for a gradually varying channel

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(Received 13 March 1978 and in revised form 25 October 1978)

Two integral invariants of Shuto's (1974) generalization of the Korteweg–de Vries equation for a unidirectional wave in a channel of gradually varying breadth b and depth d are derived. The second-order (in amplitude) invariant measures energy, as expected, but the first-order invariant measures mass divided by $b^{\frac{1}{2}}d^{\frac{1}{4}}$; accordingly, mass is conserved only if either the mean free-surface displacement vanishes or $bd^{\frac{1}{2}}$ is constant. This difficulty is associated with the reflected wave that is excited by the channel variation but neglected in the KdV approximation. The total mass flux is resolved into a primary (KdV) flux and a residual flux that is proportional to the mean displacement of the primary wave. The reflected wave associated with the residual flux is constructed by neglecting both nonlinearity and dispersion (even though both are significant for the primary wave). The results are applied to a slowly varying cnoidal wave, which is fully determined by conservation of mass and energy and the known results for a uniform channel, and to a slowly varying solitary wave, for which mass is not conserved and both trailing and reflected residuals are excited. The development of the Boussinesq equations for a gradually varying channel and their reduction to Shuto's equation are sketched in an appendix.

1. Introduction

The free-surface displacement $\eta(s,x)$ of a weakly nonlinear, weakly dispersive, unidirectional wave in a channel of gradually varying breadth $b(x)$ and depth $d(x)$ satisfies the generalized Korteweg–de Vries (KdV) equation (Shuto 1974)†

$$\tfrac{1}{3}(d^2/c^3)\,\eta_{sss}+3(cd)^{-1}\eta\eta_s+2\eta_x+(\Lambda bc)\,\eta = 0, \tag{1.1}$$

where

$$s = \int\frac{dx}{c(x)}-t \quad (c^2 \equiv gd), \tag{1.2}$$

Λ is a logarithmic differentiation operator, such that

$$\Lambda f(x) \equiv (d/dx)\log f, \tag{1.3}$$

for any function $f(x)$, and subscripts imply partial differentiation. [Shuto derives (1.1) directly from the equations of motion through a perturbation analysis. I sketch an alternative derivation, starting from the two-dimensional Boussinesq equations, in

† The generalized KdV equation (1.1) also is valid for axisymmetric wave propagation if x is taken to be the cylindrical radius and $b \equiv x$ (Miles 1978) and is relevant for geometrical-optics approximations of Whitham's type in the Boussinesq regime (cf. Ostrovskiy & Shrira 1976; Miles 1977*a*).

0022-1120/79/4217-5970 $02.00

the appendix.] The implicit scaling restrictions for a wave of amplitude a and characteristic length l are

$$a/d, d^2/l^2, l\Lambda b, l\Lambda d = O(\epsilon) \quad (\epsilon \to 0), \tag{1.4}$$

where ϵ is an appropriate expansion parameter (e.g. the maximum value of a/d). I consider here the implications of the integral invariants associated with (1.1). There are two such invariants, of first and second order in the amplitude; the latter measures energy, as expected, but the former measures mass only if $bd^{\frac{1}{2}}$ is constant (see § 2). It follows that the solutions of (1.1) conserve energy but do not, in general, conserve mass. [If $bd^{\frac{9}{4}}$ is constant, the transformation $\eta_* = \mathrm{d}^{-2}\eta$, $x_* = \int d^{\frac{1}{2}} dx$ reduces (1.1) to the KdV equation, which admits an infinite number of integral invariants; nevertheless, its solutions do not conserve mass.]

The invocation of energy invariance in the present context goes back to Boussinesq (1872) and, in a closely related context, to Rayleigh's (1876) derivation of Green's law for long waves of small amplitude a and speed $c = (gd)^{\frac{1}{2}}$:

$$a \propto b^{-\frac{1}{2}} d^{-\frac{1}{4}}. \tag{1.5}$$

[Green's original derivation of (1.5) follows from the neglect of the first two terms, i.e. of dispersion and nonlinearity, in (1.1), which then admits the general solution $\eta = (bc)^{-\frac{1}{2}} f(s)$, which implies (1.5).] Curiously, neither Rayleigh nor Lamb (1932, § 185) remarks that the invariance of energy, which is proportional to a^2bc in the Green's-law regime, is inconsistent with the invariance of mass, which is proportional to abc, unless a, and therefore $bd^{\frac{1}{2}}$, is constant. This difficulty, which *was* noticed by Boussinesq (1872) for a solitary wave in a channel of varying depth, is a consequence of the implicit neglect of the weak reflexion that accompanies the gradual variation of the channel: the reflected energy is of higher order in some appropriate measure of the channel variation and therefore has no significant effect on the wave, whereas the reflected mass is of first order and has a cumulative effect. The difficulty may be avoided for a wave that is either periodic (see § 3) or of compact support simply by choosing a horizontal reference plane such that the mean value of the free-surface displacement vanishes identically, but this strategem fails for an aperiodic disturbance of indefinite extent such as a solitary wave (see § 4).

2. Mass, momentum and energy

I assume that η either vanishes in the limits $s \to \pm\infty$ or is periodic but, for simplicity, display explicit, general results only for the former case.

The vertically averaged horizontal velocity in the wave may be approximated by (but see below)

$$u = c\{(\eta/d) + O(\epsilon)\}, \tag{2.1}$$

whilst the vertical velocity is $O(\epsilon^{\frac{1}{2}}u)$. The mass, momentum and energy (note that $du^2 \doteqdot g\eta^2$ and $dx = cds$) of the wave therefore are given by

$$M = \rho bc \int_{-\infty}^{\infty} \eta ds, \quad \mathscr{M} = \rho bcd \int_{-\infty}^{\infty} uds = Mc, \quad \mathscr{E} = \rho gbc \int_{-\infty}^{\infty} \eta^2 ds, \tag{2.2a, b, c}$$

within $1 + O(\epsilon)$. The limits of integration may be replaced by $\pm\frac{1}{2}T$ for a wave of period T.

Multiplication of (1.1) by $\frac{1}{2}(bc)^{\frac{1}{2}}$ and $bc\eta$, respectively, followed by integration over $-\infty < s < \infty$ on the assumption that η, η_s and η_{ss} vanish in the limits, yields the integral invariants

$$I = (bc)^{\frac{1}{2}}\int_{-\infty}^{\infty}\eta\, ds, \quad J = bc\int_{-\infty}^{\infty}\eta^2 ds. \tag{2.3a,b}$$

[Johnson (1973) gives the equivalents of I and J for constant breadth but identifies them with mass and momentum, respectively.] It follows that $\mathscr{E} = \rho g J$ is conserved. On the other hand,

$$M = \rho I(bc)^{\frac{1}{2}}, \quad \mathscr{M} = \rho I(bc^3)^{\frac{1}{2}}, \tag{2.4a,b}$$

so that, except for special combinations of b and d, M and $\mathscr{M}$ are conserved only if

$$\int_{-\infty}^{\infty}\eta\, ds = 0.$$

Non-conservation of momentum may be due at least partially to the horizontal thrust exerted on the fluid by the bottom and walls of the channel, but the change in M (and, in general, part of the change in $\mathscr{M}$) must be compensated by a reflected wave that is neglected in the preceding formulation.

Mass balance

I now examine the mass balance in more detail, starting from the continuity equation for the volumetric flux in the channel in the form†

$$Q \equiv bdu = \int_x^{\infty} b\eta_t\, dx. \tag{2.5}$$

It is expedient to resolve the integral into two parts through the substitution

$$\eta_t \equiv (\partial\eta/\partial t)_x = -(\partial\eta/\partial s)_x = -c\{(\partial\eta/\partial x)_t - (\partial\eta/\partial x)_s\}, \tag{2.6}$$

where s is given by (1.2). Integration by parts then yields

$$-\int_x^{\infty} bc(\partial\eta/\partial x)_t\, dx = bc\eta + \int_x^{\infty}(bc)'\,\eta\, dx \tag{2.7a}$$

(where the prime indicates d/dx) and, after the substitution of $(\partial\eta/\partial x)_s$ from (1.1) and of $dx = c\,ds$,

$$\begin{aligned}\int_x^{\infty} bc(\partial\eta/\partial x)_s\, dx &= -\int_x^{\infty}\{\tfrac{1}{6}(bd^2/c^2)\,\eta_{sss} + \tfrac{3}{2}(b/d)\,\eta\eta_s + \tfrac{1}{2}(bc)'\,\eta\}\, dx \\ &= \tfrac{1}{6}(bd^2/c)\eta_{ss} + \tfrac{3}{4}(bc/d)\,\eta^2 + \int_x^{\infty}\{\tfrac{1}{6}(bd^2/c)'\,\eta_{ss} + \tfrac{3}{4}(bc/d)'\,\eta^2 - \tfrac{1}{2}(bc)'\,\eta\}\, dx. \end{aligned} \tag{2.7b}$$

The first and second terms in the integrand of (2.7b) are uniformly $O(\epsilon)$ relative to $(bc)'\,\eta$; accordingly,

$$Q = bc\eta + \left\{\tfrac{1}{2}\int_x^{\infty}(bc)'\,\eta\, dx + \tfrac{1}{6}(bd^2/c)\eta_{ss} + \tfrac{3}{4}(bc/d)\,\eta^2\right\} \equiv Q_1 + Q_2 \tag{2.8}$$

† I have used x as both a variable and a limit of integration in (2.5) and in subsequent integrals in which the spatial argument of the integrand is implicitly the variable of integration.

is a uniformly valid first approximation to the total volumetric flux, in which $Q_1 \equiv bc\eta$ is the primary flux and Q_2 is the residual flux. $Q_2 = O(\epsilon Q_1)$ if $\eta = O(1)$, in which domain (2.8) is equivalent to (2.1), but the integral in Q_2 dominates both Q_1 and the remaining residual terms as $x \to -\infty$ if $\eta \to 0$.

I proceed on the assumption that $\eta \to 0$ as $x \to \pm\infty$ (in consequence of which the remainder of this section does not apply to periodic waves); then the dominant contributions to Q_2 are derived from the neighbourhood of the wave, say $x \doteqdot x_1(t) + O(l)$, and

$$Q \sim \tfrac{1}{2}[(bc)' c]_1 \int_{-\infty}^{\infty} \eta ds = \tfrac{1}{2} I[(c/b)^{\frac{1}{2}}(bc)']_1 \equiv \mathscr{Q}(x_1) \quad (x_1 - x \ll l), \tag{2.9}$$

where the subscript 1 implies $x = x_1$. Differentiation of (2.4*a*) with I constant and comparison with (2.9) then confirms the mass balance

$$c(dM/dx) = \rho \mathscr{Q}. \tag{2.10}$$

Reflected wave

The reflected wave that is necessary to compensate for the residual flux $\mathscr{Q}$, but which is neglected in the derivation of (1.1), may be approximated by assuming that its length scale is that of the channel variation, $L \gg l$, and that its amplitude is small compared with that of the primary wave. Both nonlinearity and dispersion then may be neglected (even though both are important for the primary wave), by virtue of which the reflected wave must be of the form (Lamb 1932, § 185)

$$\eta_- = (bc)^{-\frac{1}{2}} f(s_-), \quad s_- = \int \frac{dx}{c} + t. \tag{2.11a, b}$$

The amplitude a_- of η_- at a distance behind the primary wave that is large compared with l but small compared with L is determined from the facts that η_- recedes from the primary wave with the relative speed $2c\{1 + O(\epsilon)\}$ and carries the residual flux $\mathscr{Q}$; it follows that $a_- b(-2c) = \mathscr{Q}$ and hence that

$$f_1 = a_-(b_1 c_1)^{\frac{1}{2}} = -\tfrac{1}{2}(b_1 c_1)^{-\frac{1}{2}} \mathscr{Q} = -\tfrac{1}{4} I[b^{-1}(bc)']_1. \tag{2.12}$$

The reflected wave at any point P in the x, t plane then is determined by (2.11*a*) and the conservation of f along the characteristic $s_- =$ constant that joins P to the trajectory of the primary wave at the point P_- (see figure 1).

The preceding results suggest that solutions of (1.1) that do not satisfy

$$\int_{-\infty}^{\infty} \eta \, ds = 0,$$

such as that for a slowly varying solitary wave (see § 4), must be regarded with some caution. It appears likely that such solutions cannot be uniformly valid as $t \uparrow \infty$, but this does not exclude the possibility that they are viable approximations in some contexts.

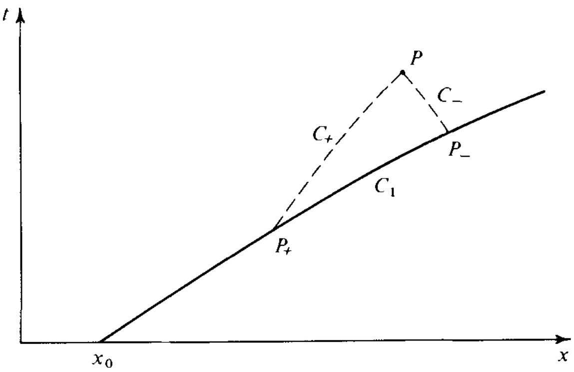

FIGURE 1. The characteristics $C_\pm$ ($s_\pm$ = constant) projected from a point P in the x, t plane to the trajectory C_1 of the primary wave. The reflected wave at P is obtained by conserving $(bc)^{\frac{1}{2}}a_-$ along C_- from P_- and multiplying the result by $(bc)^{-\frac{1}{2}}$ [see (2.11) and (2.12)]. The trailing residual at P (see § 4) is similarly obtained by conserving $(bc)^{\frac{1}{2}}a_+$ along C_+.

3. Slowly varying cnoidal wave

The solution of (1.1) for waves of prescribed period $T \equiv (L/g)^{\frac{1}{2}}$ and length

$$l = cT = (dL)^{\frac{1}{2}} \tag{3.1}$$

may be approximated by that for a cnoidal wave in water of constant depth (Lamb 1932, § 253) in the form

$$\eta = a(x)\, N(\theta, x), \quad \theta = \frac{1}{T}\{s - \tau(x)\} \equiv \frac{1}{T}\left\{\int \frac{(1-\gamma)\,dx}{c} - t\right\}, \tag{3.2a, b}$$

$$N = \mathrm{cn}^2(2K\theta|m) - \langle \mathrm{cn}^2 \rangle, \quad \langle \mathrm{cn}^2 \rangle = \{m - 1 + (E/K)\}/m, \tag{3.3a, b}$$

$$al^2/d^3 = aL/d^2 = \tfrac{16}{3} m K^2 \equiv \mathscr{U}(m), \tag{3.4}$$

and

$$\gamma = [\{2 - m - 3(E/K)\}/(2m)]\,(a/d), \tag{3.5}$$

where θ and x are fast and slow variables, a is a slowly varying amplitude, τ/T is a slowly varying phase shift, $c/(1-\gamma)$ is the phase speed (with which an observer must move to conserve θ), $\mathrm{cn}(u|m)$ is an elliptic cosine of slowly varying modulus $\sqrt{m}$, K and E are complete elliptic integrals in the notation of Abramowitz & Stegun (1965, p. 587), $\mathscr{U}(m)$ is the local Ursell parameter, and the restrictions $l\Lambda b$, $l\Lambda d = O(\epsilon)$ of §§ 1 and 2 now are replaced by $|l\Lambda b|$, $|l\Lambda d| \ll \epsilon$.

The integral invariants obtained by substituting (3.2) into (2.3) and replacing the limits of integration by $\pm\frac{1}{2}T$ are

$$I = a(bc)^{\frac{1}{2}}\, T\, \langle N \rangle, \quad J = a^2 bcT\, \langle N^2 \rangle, \tag{3.6a, b}$$

where angular brackets imply an average over a unit interval of θ. It follows from (3.3) that $\langle N \rangle = 0$ (a necessary condition for the joint conservation of mass and energy unless both b and d are constant), whilst (3.6b) implies the constraint

$$JL^{\frac{3}{2}}/(bd^{\frac{9}{2}}) = \mathscr{U}^2 \langle N^2 \rangle \equiv \mathscr{F}(m), \tag{3.7}$$

where

$$\mathscr{F} = (4^4/3^3)\, K^2\{2(2-m)\, EK - 3E^2 - (1-m)\, K^2\}. \tag{3.8}$$

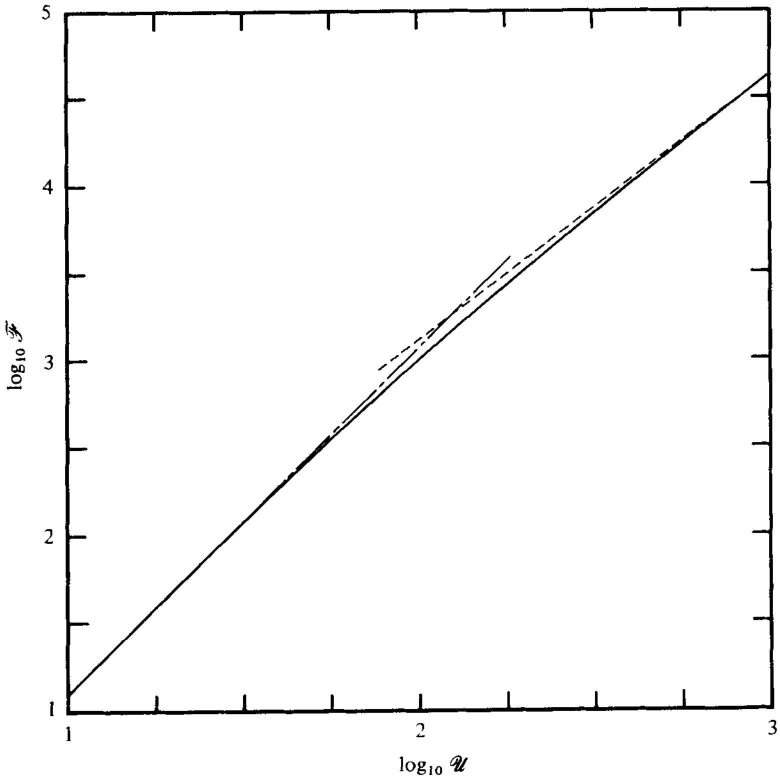

FIGURE 2. $\text{Log}_{10}\mathscr{F}$ *vs.* $\log_{10}\mathscr{U}$, as determined from (3.4) and (3.8) (——), (3.9a) (–––) and (3.10a) (- - -). The evolution of the amplitude from a prescribed amplitude a_0 at a particular station, at which $b = b_0$ and $d = d_0$, is obtained by measuring

$$\log\mathscr{F} - \log\mathscr{F}_0 = \log(b_0 d_0^{\frac{9}{4}}) - \log(bd^{\frac{9}{4}}) \; vs. \log\mathscr{U} - \log\mathscr{U}_0 = \log(a/d^2) - \log(a_0/d_0^2)$$

from the point determined by $\mathscr{U} = \mathscr{U}_0$.

It follows from (3.7), which determines $m(x)$, that m is constant if and only if $bd^{\frac{9}{4}}$ = constant, in which special case (3.2)–(3.5) describe an exact similarity solution of (1.1). In general, (3.3a) is the first term in an asymptotic expansion; however, approximations of this type (which conserve mass, momentum and energy) are often rather better than formal asymptotic considerations would suggest.

The results (3.4) and (3.7) provide a parametric relation between aL/d^2 and $JL^{\frac{3}{2}}/bd^{\frac{9}{4}}$ that may be graphically represented as a plot of $\log\mathscr{U}$ *vs.* $\log\mathscr{F}$ (figure 2). The limiting relations

$$\mathscr{F} \to \tfrac{1}{8}\mathscr{U}^2, \quad a \to (8J)^{\frac{1}{2}} b^{-\frac{1}{2}}(Ld)^{-\frac{1}{4}} \quad (\mathscr{U} \downarrow 0) \tag{3.9a,b}$$

and

$$\mathscr{F} \sim (\tfrac{4}{3}\mathscr{U})^{\frac{3}{2}}, \quad a \sim \tfrac{3}{4}J^{\frac{2}{3}}b^{-\frac{2}{3}}d^{-1} \quad (\mathscr{U} \uparrow \infty) \tag{3.10a,b}$$

intersect at $\mathscr{U} \doteqdot 150$ (see figure 2). Moreover, (3.9), which corresponds to Green's law, is in error by less than 1 % for $\mathscr{U} < 20$, whilst

$$\mathscr{F} \sim (\tfrac{4}{3}\mathscr{U})^{\frac{3}{2}}\{1 - 2(\tfrac{1}{3}\mathscr{U})^{-\frac{1}{2}}\} \quad (\mathscr{U} \uparrow \infty), \tag{3.11}$$

which neglects only terms of exponentially small order, is in error by less than 1 % for $\mathscr{U} > 70$. The case of constant depth is especially simple in that the plot of $\log\mathscr{F}$ *vs.* $\log\mathscr{U}$ is equivalent to $-\log b$ *vs.* $\log a$.

The present problem has been considered by Ostrovskiy & Pelinovskiy (1975) and Shuto (1974) and by Ostrovskiy & Pelinovskiy (1970) and Svendsen & Brink-Kjaer (1972) for b = constant. Svendsen & Brink-Kjaer replace the integral in (3.2b) by $(1-\gamma)(x/l)$, which is clearly incorrect; however, this is evidently a minor slip and does not affect their results for amplitude, which are more complete than those of Ostrovskiy & Pelinovskiy (1970). Shuto allows for the variation of both breadth and depth and states a differential equation, his (56), that appears to be analytically intractable and is inconsistent with conservation of energy except in the limits $\mathscr{U} \downarrow 0$ and $\mathscr{U} \uparrow \infty$, in which it is consistent with (3.9) and (3.11); nevertheless, his graphical results are not significantly in error over the entire range of $\mathscr{U}$. Ostrovskiy & Pelinovskiy (1975) state equivalents of (3.4) and (3.7), but their results are less complete than those given here.

Shuto (1974) compares his results with his own experimental observations and with those of Iwagagi & Sakai (1969) for shoaling waves with periods from 1·2 to 6 s on uniform slopes of $\frac{1}{20}$ and $\frac{1}{70}$. He concludes that linear surface-wave theory (which presumably accounts exactly for dispersion) is superior to his cnoidal-wave results for $\mathscr{U} < 30$ and conversely for $\mathscr{U} > 30$ and that the latter are good for a/d as large as 0·8.

Svendsen & Hansen (1978) obtain the next term in the asymptotic expansion of N for b = constant and report 'good agreement' between predicted and observed wave profiles 'even for ... waves rather close to breaking'.

4. Slowly varying solitary wave

Rescaling θ in (3.2) and letting $\mathscr{U} \uparrow \infty$ $(T, L \uparrow \infty)$ with J fixed in (3.3)–(3.7) yields the slowly varying solitary wave

$$\eta = a(x)\,\mathrm{sech}^2\,[\omega(x)\{s-\tau(x)\}], \tag{4.1}$$

where

$$a = \tfrac{3}{4}J^{\frac{2}{3}}b^{-\frac{2}{3}}d^{-1} = a_0\mathscr{b}^{-\frac{2}{3}}\mathscr{d}^{-1}, \tag{4.2}$$

$$\omega = (3ga)^{\frac{1}{2}}(2d)^{-1} = \omega_0\mathscr{b}^{-\frac{1}{3}}\mathscr{d}^{-\frac{3}{2}}, \tag{4.3}$$

$$\tau(x) = \frac{1}{2}\int_{x_0}^{x}\left(\frac{a}{d}\right)\frac{dx}{c} = \frac{1}{2}\left(\frac{a_0}{c_0 d_0}\right)\int_{x_0}^{x}\mathscr{b}^{-\frac{2}{3}}\mathscr{d}^{-\frac{5}{2}}\,dx, \tag{4.4}$$

$$\mathscr{b} = b/b_0, \quad \mathscr{d} = d/d_0, \tag{4.5a,b}$$

the subscript zero implies $x = x_0$, and the constant of integration in (1.2) now is implicitly chosen to yield $s = 0$ at $x = x_0$ and $t = 0$. The prediction that $a \propto d^{-1}$ for constant b is due to Boussinesq (1872). The prediction that $a \propto b^{-\frac{2}{3}}d^{-1}$ appears to be due originally to Saeki, Takagi & Ozaki (1971, cited by Shuto 1974); see also Shuto (1974), Ostrovskiy & Pelinovskiy (1975) and Miles (1977b). Shuto (1973) cites experimental results that support the prediction $a \propto d^{-1}$ for constant b and sufficiently small d', but the data are inadequate for a firm delineation of the parametric regime in an a/d, d' plane. Experiments by Chang & Melville (unpublished) support the prediction $a \propto b^{-\frac{2}{3}}$ for an expanding channel of linearly varying breadth and constant depth.

Substituting (4.1) into (2.3a) and (2.2a) yields

$$I = 2J^{\frac{1}{3}}g^{-\frac{1}{4}}b^{\frac{1}{6}}d^{\frac{3}{4}} = I_0\mathscr{b}^{\frac{1}{6}}\mathscr{d}^{\frac{3}{4}}, \quad I_0 = 4(\tfrac{1}{3}a_0b_0)^{\frac{1}{2}}c_0^{-\frac{1}{2}}d_0^{\frac{3}{2}} \tag{4.6a,b}$$

and

$$M = 2\rho J^{\frac{1}{3}}b^{\frac{2}{3}}d = M_0\mathscr{b}^{\frac{2}{3}}\mathscr{d}, \quad M_0 = 4\rho(\tfrac{1}{3}a_0)^{\frac{1}{2}}b_0d_0^{\frac{3}{2}}. \tag{4.7a,b}$$

The integral I is conserved if and only if $bd^{\frac{9}{4}} =$ constant, in which special case (4.1) is an exact similarity solution of (1.1); however, this similarity solution does not conserve mass. Mass is conserved if and only if $bd^{\frac{3}{2}} =$ constant. [The failure of the condition $\langle N \rangle = 0$ in the limit $\mathscr{U} \uparrow \infty$ is a consequence of the loss of the displacement

$$-a\langle \mathrm{cn}^2 \rangle \sim a/K,$$

which cancels the mean of $a\mathrm{cn}^2(2K\theta)$ when integrated over $-K < 2K\theta < K$.] The logarithmic derivative of the mass flux may be resolved into components associated with the variation of I and with the neglect of the reflected wave according to

$$\Lambda M = \Lambda I + \tfrac{1}{2}\Lambda bc = (\tfrac{1}{6}\Lambda b + \tfrac{3}{4}\Lambda d) + (\tfrac{1}{2}\Lambda b + \tfrac{1}{4}\Lambda d). \tag{4.8a, b}$$

The fact that I [see (4.6)] is not constant suggests that (4.1) is not a uniformly valid approximation to the solution of (1.1), but rather that it is the first term in an inner expansion. It is relatively straightforward to construct higher-order terms in this inner expansion (cf. Johnson 1973; Ko & Kuehl 1978), but the results are ambiguous in the absence of a matched outer expansion, the construction of which poses significant difficulties (cf. Johnson 1973). An alternative procedure, which avoids some of the difficulties associated with matched asymptotic expansions, for improving the approximation (4.1) is to construct a perturbation solution of (1.1) through a perturbation of the inverse-scattering solution of the KdV equation (Karpman & Maslov 1977; Kaup & Newell 1978).

Either of the foregoing procedures should render I invariant, but both neglect the reflected wave, in consequence of which neither renders M invariant. A simpler, albeit more *ad hoc*, procedure that conserves both I and M is to posit

$$\eta = \eta_1 + \eta_+ + \eta_-, \tag{4.9}$$

where η_1 is the primary wave given by (4.1);

$$\eta_\pm = (bc)^{-\frac{1}{2}} f_\pm(s_\pm), \quad s_\pm = \mp t + \int_{x_0}^{x} \frac{dx}{c} \tag{4.10a, b$)_\pm$}$$

(subscripts are vertically ordered), is a right/left-moving secondary wave for which the length scale is that of the channel variation [cf. (2.11)]; $\eta_1 + \eta_+$ is a perturbation solution of (4.1) that conserves I; $\eta_1 + \eta_+ + \eta_-$ is a perturbation solution of the Boussinesq equations that conserves M. Assume, for definiteness, that the channel is uniform in $x < x_0$. The domain of η_+ then is $0 < s_+ < \tau(x)$, and

$$\frac{d}{dx}\left\{ I_1 + \int_0^{\tau(x)} f_+(s)\, ds \right\} = 0, \tag{4.11}$$

where I_1 is given by (4.6a). It follows that

$$f_+(\tau) = -I_1'/\tau' \equiv (bc)^{\frac{1}{2}} a_+, \tag{4.12a}$$

where

$$a_+ = -4 \times 3^{-\frac{3}{2}} a^{-\frac{1}{2}} d^{\frac{5}{2}} \Lambda(bd^{\frac{9}{4}}) \tag{4.12b}$$

is the amplitude of η_+ just behind the primary wave. The secondary wave η_+ at any point P in the x, t plane then is determined by (4.10a)$_+$ and the conservation of f_+ along the characteristic $s_+ =$ constant that joins P to the point P_+ on the trajectory of the primary wave (see figure 1). The conserved (by $\eta_1 + \eta_+$) value of I is I_0 [see (4.6b)].

The corresponding calculation of η_- has already been carried out in § 2. Substituting $I = I_0$ from (4.6*b*) into (2.12) and altering the notation to parallel that of (4.12) yields

$$f_- = (bc)^{\frac{1}{2}} a_- \quad \text{on} \quad s_- = 2\int_{x_0}^{x} \frac{dx}{c} - \tau, \tag{4.13a,b}$$

where

$$a_- = -(\tfrac{1}{3}a_0)^{\frac{1}{2}} d_0^{\frac{3}{2}} \mathscr{b}^{-\frac{1}{2}} \mathscr{d}^{\frac{1}{4}} \Lambda(bd^{\frac{1}{2}}). \tag{4.13c}$$

The amplitude a_+ agrees with that of the 'shelf' calculated from the formulation of either Ko & Kuehl (1978) or Kaup & Newell (1978) at a point that is far enough behind the primary wave to permit the approximation $\tanh[\omega(s-\tau)] \doteqdot -1$ but sufficiently close to permit bc to be approximated by its local value for the primary wave. Ko & Kuehl's formulation is invalid at more distant points, but Kaup & Newell's formulation presumably is uniformly valid in a more extended regime.

The approximation (4.4) may not provide an adequate approximation to the gradually varying phase $\tau(x)$, and it therefore is worth noting that the formulations of Ko & Kuehl and Kaup & Newell both yield the second approximation

$$\tau = \tau_1 + \tfrac{1}{2}(\omega^{-1} - \omega_0^{-1}), \tag{4.14}$$

where τ_1, ω and ω_0 are given by (4.3) and (4.4). The corresponding correction to the argument of the hyperbolic secant in (4.1) is $\frac{1}{2}(1 - \mathscr{b}^{-\frac{1}{3}} \mathscr{d}^{-\frac{3}{2}})$, which may have a significant effect on the trajectory of the peak of the primary wave.

The results in this section are qualitatively supported by the numerical calculations of Maxon & Viecelli (1974) for unidirectional, spherically symmetric solitons. The hypotheses of Boussinesq similarity and conservation of energy imply that the amplitude of the spherical soliton should vary like $x^{-\frac{4}{3}}$. On applying this prediction to the results in their figure 1, I find that it is confirmed within the accuracy of the data. They also obtain a 'small residue' that has only a relatively negligible energy and appears to be the counterpart of η_+.

This work was partially supported by the Physical Oceanography Division, National Science Foundation, NSF Grant OCE74-23791, and by the Office of Naval Research under Contract N00014-76-C-0025. The use of linearized characteristics theory to calculate the reflected wave η_- was suggested by Peregrine's (1967) treatment of a solitary wave on a uniformly sloping beach.

Appendix. Boussinesq equations

The Boussinesq equations for waves in a channel of gradually varying breadth $b(x)$ and depth $d(x)$ can be obtained through a straightforward generalization of Whitham's (1967) derivation of the Boussinesq equations for two-dimensional waves. His Lagrangian density (9) may be shown to be valid for gradually varying depth, and the assumption that it is independent of the transverse co-ordinate followed by the invocation of the differential metric $\{dx, b(x)\,dx\}$ leads to

$$\tfrac{1}{3}(bd^3\xi_{xx})_{xx} + \{b(d+\eta)\,\xi_x\}_x + b\eta_t = 0 \tag{A 1a}$$

and

$$\xi_t + \tfrac{1}{2}\xi_x^2 + g\eta = 0, \tag{A 1b}$$

where ξ and η are the velocity potential at and the displacement of the free surface, and subscripts imply partial differentiation [(A 1*a,b*) are the counterparts of Whitham's

(12) with $\xi = F$, $\eta = h - h_0$ and $d = h_0$ in his notation]. The implicit scaling restrictions for a wave of amplitude a and length l are equivalent to those of (1.4).

The KdV equation (1.1) may be deduced from (A 1) by eliminating η, introducing s from (1.2), assuming that $|\xi_s| \ll |\xi_x|$, and then letting $\xi_s \doteqdot g\eta$ in the end result.

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